Nonconservative Noether's Theorem in Optimal Control*

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Abstract

We extend Noether's theorem to dynamical optimal control systems being under the action of nonconservative forces. A systematic way of calculating conservation laws for nonconservative optimal control problems is given. As a corollary, the conserved quantities previously obtained in the literature for nonconservative problems of mechanics and the calculus of variations are derived.

Keywords: Nonconservative forces, symmetry, conservation laws, Noether's theorem, optimal control.

2000 Mathematics Subject Classification: 49K05, 49K15, 49S05, 70H33, 37J15, 93C10.

1 Introduction

The concept of symmetry plays an important role both in Physics and Mathematics. Symmetries are described by transformations of the system, which result in the same object after the transformation is carried out. They are described mathematically by parameter groups of transformations.

*To be presented at *13th IFAC Workshop on Control Applications of Optimisation*, 26-28 April 2006, Paris - Cachan, France. Accepted (19-12-2005) for the Proceedings, IFAC publication, Elsevier Ltd, Oxford, UK. Research Report CM05/I-54.

Their importance ranges from fundamental and theoretical aspects to concrete applications, having profound implications in the dynamical behavior of the systems, and in their basic qualitative properties.

Another fundamental notion in Physics and Mathematics is the one of conservation law. Typical application of conservation laws in the calculus of variations and optimal control is to reduce the number of degrees of freedom, and thus reducing the problems to a lower dimension, facilitating the integration of the differential equations given by the necessary optimality conditions.

Emmy Noether was the first who proved, in 1918, that the notions of symmetry and conservation law are connected: when a system exhibits a symmetry, then a conservation law can be obtained. One of the most important and well known illustrations of this deep and rich relation, is given by the conservation of energy in Mechanics: the autonomous Lagrangian $L(q, \dot{q})$, correspondent to a mechanical system of conservative points, is invariant under time-translations (time-homogeneity symmetry), and

$$-L\left(q(t), \dot{q}(t)\right) + \frac{\partial L}{\partial \dot{q}}\left(q(t), \dot{q}(t)\right) \cdot \dot{q}(t) \equiv \text{constant} \tag{1}$$

follows from Noether's theorem, i.e., the total energy of a conservative closed system always remain constant in time, "it cannot be created or destroyed, but only transferred from one form into another". Expression (1) is valid along all the Euler-Lagrange extremals $q(\cdot)$ of an autonomous problem of the calculus of variations. The conservation law (1) is known in the calculus of variations as the 2nd Erdmann necessary condition; in concrete applications, it gains different interpretations: conservation of energy in Mechanics; income-wealth law in Economics; first law of Thermodynamics; etc. The literature on Noether's theorem is vast, and many extensions of the classical results of Emmy Noether are now available for the more general setting of optimal control (see e.g. [2, 5, 10, 11], and references therein). Here we remark that in all those results conservation laws always refer to closed systems.

It turns out that in practical terms closed systems do not exist: forces that do not store energy, so-called nonconservative or dissipative forces, are always present in real systems. Friction is a nonconservative force, but others do exist. Any friction-type force, like air resistance, is a nonconservative force. Nonconservative forces remove energy from the systems and, as a consequence, the conservation law (1) is broken. This explains, for instance, why the innumerable "perpetual motion machines" that have been proposed fail. In presence of external nonconservative forces, Noether's theorem and

respective conservation laws cease to be valid. However, it is still possible to obtain a Noether-type theorem which covers both conservative (closed system) and nonconservative cases [3, 4]. Roughly speaking, one can prove that Noether's conservation laws are still valid if a new term, involving the nonconservative forces, is added to the standard conservation laws.

Here we extend previous nonconservative results [3, 4] to the wider context of optimal control. For that, and differently from [3, 4], where the Lagrangian formalism is considered, we adopt an Hamiltonian point of view.

2 Preliminaries

Let us consider the optimal control problem in Lagrange form:

$$I[q(\cdot),u(\cdot)] = \int_a^b L\left(t,q(t),u(t)\right) dt \longrightarrow \min\,, \tag{P}$$
$$\dot{q}(t) = \varphi\left(t,q(t),u(t)\right)\,,$$

together with some boundary conditions on $q(\cdot)$. In problem (P) $\dot{q} = \frac{dq}{dt}$, and the Lagrangian $L : [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and the velocity vector $\varphi : [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are assumed to be C^1 functions with respect to all the arguments. In agreement with the calculus of variations, we assume the admissible state trajectories to be piecewise smooth, and the admissible control functions to be piecewise constant with no restrictions on their values: $q(\cdot) \in PC^1([a,b];\mathbb{R}^n)$, $u(\cdot) \in PC([a,b];\mathbb{R}^m)$.

Remark 1 *There is no classical Hamiltonian theory for constrained variational problems. For an ongoing attempt to develop it, we refer the reader to [1, Ch. 6].*

Remark 2 *The fundamental problem of the calculus of variations,*

$$I[q(\cdot)] = \int_a^b L\left(t,q(t),\dot{q}(t)\right) \longrightarrow \min\,, \tag{2}$$

is a particular case of problem (P)*: in that case $\varphi(t,q,u) = u$. The problems of the calculus of variations with higher-order derivatives are also easily written in the optimal control form* (P)*. For example, the problem of the calculus of variations with derivatives of second order,*

$$I[q(\cdot)] = \int_a^b L\left(t,q(t),\dot{q}(t),\ddot{q}(t)\right) \longrightarrow \min\,, \tag{3}$$

is equivalent to problem

$$I[q^0(\cdot), q^1(\cdot), u(\cdot)] = \int_a^b L\left(t, q^0(t), q^1(t), u(t)\right) \longrightarrow \min\,,$$
$$\begin{cases} \dot{q}^0(t) = q^1(t)\,, \\ \dot{q}^1(t) = u(t)\,. \end{cases}$$

In the fifties of the twentieth century, L.S. Pontryagin and his collaborators proved the main necessary optimality condition for optimal control problems: the famous Pontryagin Maximum Principle [9].

Definition 3 (Process) *An admissible pair* $(q(\cdot), u(\cdot))$ *satisfying the control system* $\dot{q}(t) = \varphi\left(t, q(t), u(t)\right)$ *of* (P), $t \in [a, b]$, *is called a* process.

Theorem 4 (Pontryagin Maximum Principle– [9]) *If* $(q(\cdot), u(\cdot))$ *is an optimal process for problem* (P)*, then there exists a vectorial function* $p(\cdot) \in PC^1([a, b]; \mathbb{R}^n)$ *such that the following conditions hold:*

- *the Hamiltonian system*

$$\begin{cases} \dot{q}(t) & = \frac{\partial \mathcal{H}}{\partial p}(t, q(t), u(t), p(t))\,, \\ \dot{p}(t) & = -\frac{\partial \mathcal{H}}{\partial q}(t, q(t), u(t), p(t))\,; \end{cases} \tag{4}$$

- *the stationary condition*

$$\frac{\partial \mathcal{H}}{\partial u}(t, q(t), u(t), p(t)) = 0\,; \tag{5}$$

with the Hamiltonian $\mathcal{H}$ *defined by*

$$\mathcal{H}\left(t, q, u, p\right) = -L\left(t, q, u\right) + p \cdot \varphi\left(t, q, u\right)\,. \tag{6}$$

Remark 5 *In mechanics,* p *corresponds to the* generalized momentum*. In the language of optimal control* p *is called the* adjoint variable.

Definition 6 *Any triplet* $(q(\cdot), u(\cdot), p(\cdot))$ *satisfying the conditions of Theorem 4 is called a* Pontryagin extremal.

Remark 7 *The Pontryagin Maximum Principle is more general than we state it here. Written as in Theorem 4, the Pontryagin Maximum Principle is also known as* Hestenes Theorem. *In particular, we are only considering normal Pontryagin extremals.*

From the Hamiltonian system (4) and the stationary condition (5), it follows that

$$\frac{d\mathcal{H}}{dt}(t,q(t),u(t),p(t)) = \frac{\partial \mathcal{H}}{\partial t}(t,q(t),u(t),p(t)) \,. \tag{7}$$

When the optimal control problem (P) is autonomous (when the Hamiltonian (6) does not depend explicitly on time t) one obtains from (7) the conservation law

$$\mathcal{H}(t,q(t),u(t),p(t)) = const \,. \tag{8}$$

For the fundamental problem of the calculus of variations (2) ($\varphi = u \Rightarrow \mathcal{H} = -L + p \cdot u$) one obtains from the Pontryagin Maximum Principle:

$$\begin{gathered}
\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = u \,, \\
\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = \frac{\partial L}{\partial q} \,, \\
\frac{\partial \mathcal{H}}{\partial u} = 0 \Leftrightarrow p = \frac{\partial L}{\partial u} \Rightarrow \dot{p} = \frac{d}{dt}\frac{\partial L}{\partial u} \,.
\end{gathered}$$

Comparing the two expressions for $\dot{p}$, one arrives to the Euler-Lagrange differential equations:

$$\frac{d}{dt}\frac{\partial L}{\partial u} = \frac{\partial L}{\partial q}. \tag{9}$$

When the fundamental problem of the calculus of variations is autonomous, equality (8) reduces to (1).

In the presence of nonconservative forces $Q(t,q(t),\dot{q}(t))$, i.e. forces which are not equivalent to the gradient of a potential, like friction and drag, the Euler-Lagrange equations (9) are no longer valid, and it is well-known that they must be substituted by

$$\frac{d}{dt}\frac{\partial L}{\partial u} = \frac{\partial L}{\partial q} + Q \tag{10}$$

(see e.g. [4]). In Physics the Hamiltonian formalism is not common for the nonconservative case. However, it is clear that Theorem 4 and property (7) must be also changed when considering the influence of external dissipative forces Q. This will be addressed in §3.

Following [2], the notion of invariance for problem (P) is defined in terms of the Hamiltonian, by introducing the augmented functional

$$J[q(\cdot),u(\cdot),p(\cdot)] = \int_a^b \left[\mathcal{H}\left(t,q(t),u(t),p(t)\right) - p(t)\cdot\dot{q}(t)\right] dt \,,$$

where $\mathcal{H}$ is given by (6).

Definition 8 (Invariance up to a gauge term – cf. [2]) *An optimal control problem* (P) *is said to be invariant under the* ε*-parameter local group of transformations*

$$\begin{cases} \bar{t}(t) = t + \varepsilon\tau(t, q(t), u(t), p(t)) + o(\varepsilon)\,, \\ \bar{q}(t) = q(t) + \varepsilon\xi(t, q(t), u(t), p(t)) + o(\varepsilon)\,, \\ \bar{u}(t) = u(t) + \varepsilon\sigma(t, q(t), u(t), p(t)) + o(\varepsilon)\,, \\ \bar{p}(t) = p(t) + \varepsilon\alpha(t, q(t), u(t), p(t)) + o(\varepsilon)\,, \end{cases} \tag{11}$$

if, and only if, there exists a function Λ *such that*

$$[\mathcal{H}(\bar{t}, \bar{q}, \bar{u}, \bar{p}) - \bar{p} \cdot \dot{\bar{q}}]\, d\bar{t} = [\mathcal{H}(t, q, u, p) - p \cdot \dot{q}]\, dt + \varepsilon d\Lambda(t, q, u, p)\,. \tag{12}$$

Remark 9 *Function* Λ *of Definition 8 is called a* gauge term *in the Physics literature. In the particular case* $\Lambda = 0$*, one obtains the concept of* absolute invariance.

Remark 10 *We can write equation* (12) *in the following way:*

$$[\mathcal{H}(\bar{t}, \bar{q}, \bar{u}, \bar{p}) - \bar{p} \cdot \dot{\bar{q}}]\frac{d\bar{t}}{dt} = [\mathcal{H}(t, q, u, p) - p \cdot \dot{q}] + \varepsilon\frac{d\Lambda}{dt}. \tag{13}$$

Functions τ, ξ, σ, and α are known as the *infinitesimal generators* of the invariance-transformations (11). Next theorem asserts that generators are sufficient to define invariance.

Theorem 11 (Necessary and sufficient condition of invariance) *Problem* (P) *is said to be invariant up to the gauge term* Λ *if, and only if,*

$$\tau\frac{\partial\mathcal{H}}{\partial t} + \xi \cdot \frac{\partial\mathcal{H}}{\partial q} + \sigma \cdot \frac{\partial\mathcal{H}}{\partial u} + \alpha \cdot (\frac{\partial\mathcal{H}}{\partial p} - \dot{q}) - \dot{\xi} \cdot p + \dot{\tau}\mathcal{H} = \frac{d\Lambda}{dt}. \tag{14}$$

Proof. Having in mind that for $\varepsilon = 0$ one has $\bar{t} = t$, $\bar{q} = q$, $\bar{u} = u$, $\bar{p} = p$, we differentiate (13) with respect to ε and then put $\varepsilon = 0$. ∎

Definition 12 *A quadruple* $(\tau, \xi, \sigma, \alpha)$ *is said to be a* symmetry *of the optimal control problem* (P) *if it satisfies condition* (14) *for a certain* Λ*. We talk about* exact symmetries *if* $\Lambda = 0$*.*

Remark 13 *It is possible to use a modern computer algebra system to compute the symmetries of an optimal control problem* (P) *in an automatic way – see [6].*

Remark 14 *For the fundamental problem of the calculus of variations* (2), *the necessary and sufficient condition of invariance* (14) *takes the well-known form (cf. e.g. [7, pp. 429])*

$$\tau \frac{\partial L}{\partial t} + \xi \cdot \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}} \cdot \left(\dot{\xi} - \dot{q}\dot{\tau}\right) + \dot{\tau} L = \frac{d\Lambda}{dt}\,.$$

Remark 15 *For the problem of the calculus of variations with derivatives of second order* (3), *the necessary and sufficient condition of invariance* (14) *takes the form (cf. [12, Lemma 5.5])*

$$\tau \frac{\partial L}{\partial t} + \xi_0 \cdot \frac{\partial L}{\partial q} + \xi_1 \cdot \frac{\partial L}{\partial \dot{q}} + \left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\right) \cdot \left(\dot{\xi}_0 - \dot{\tau}\dot{q}\right) + \frac{\partial L}{\partial \ddot{q}} \cdot \left(\dot{\xi}_1 - \dot{\tau}\ddot{q}\right) + \dot{\tau} L = \frac{d\Lambda}{dt}\,.$$

3 Main Result

We begin by introducing the notion of nonconservative Hamiltonian system. Such concept must lead us to equations (10) in the particular case of the fundamental problem of the calculus of variations under presence of nonconservative forces Q.

Definition 16 (Nonconservative Hamiltonian System) *We define the* nonconservative Hamiltonian system *by*

$$\begin{cases} \dot{q}(t) &= \frac{\partial \mathcal{H}}{\partial p}(t, q(t), u(t), p(t))\,, \\ \dot{p}(t) &= -\frac{\partial \mathcal{H}}{\partial q}(t, q(t), u(t), p(t)) + Q(t, q(t), u(t))\,, \end{cases} \tag{15}$$

where $\mathcal{H}$ is given as in (6).

Remark 17 *For the particular case of the fundamental problem of the calculus of variations ($\varphi = u$), the nonconservative Hamiltonian system* (15), *together with the stationary condition* (5), *lead us to the nonconservative Euler-Lagrange equations* (10):

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = u\,,$$
$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} + Q = \frac{\partial L}{\partial q} + Q\,,$$
$$\frac{\partial \mathcal{H}}{\partial u} = 0 \Leftrightarrow p = \frac{\partial L}{\partial u} \Rightarrow \dot{p} = \frac{d}{dt}\frac{\partial L}{\partial u}\,,$$

and comparing the two expressions for $\dot{p}$ we obtain equation (10).

Similarly to Definition 6, we introduce now the notion of nonconservative extremal.

Definition 18 (Nonconservative Extremal) *Any triplet* $(q(\cdot), u(\cdot), p(\cdot))$*, satisfying the stationary condition* (5) *and the nonconservative Hamiltonian system* (15)*, will be called a* nonconservative extremal*.*

Proposition 19 *The following property holds along the nonconservative extremals:*

$$\frac{d\mathcal{H}}{dt}(t, q(t), u(t), p(t)) \\ = \frac{\partial \mathcal{H}}{\partial t}(t, q(t), u(t), p(t)) + Q(t, q(t), u(t)) \cdot \frac{\partial \mathcal{H}}{\partial p}(t, q(t), u(t), p(t)) \,. \quad (16)$$

Proof. Equality (16) is a simple consequence of the the stationary condition (5) and the nonconservative Hamiltonian system (15). ■

Remark 20 *In the particular case* $Q = 0$ *(conservative case), the nonconservative Hamiltonian system* (15) *takes the form* (4)*, and the set of nonconservative extremals coincide with the set of Pontryagin extremals. In that situation, property* (16) *equals* (7)*.*

Definition 21 (Nonconservative Constants of Motion) *We say that a function* $C(t, q, u, p)$ *is a* nonconservative constant of motion *if it is preserved along any nonconservative extremal* $(q(\cdot), u(\cdot), p(\cdot))$*:*

$$C(t, q(t), u(t), p(t)) = c \,, \quad c \text{ constant} \;, \quad \forall\, t \in [a, b] \,. \quad (17)$$

Equation (17) *is then said to be a* nonconservative conservation law*.*

Generalizations of the classical Noether's theorem of the calculus of variations include: (i) (conservative) Noether-type theorems for the more general framework of optimal control, asserting the existence of a preserved quantity along the Pontryagin extremals, whenever a symmetry occurs – see [2, 5, 10, 11]; (ii) Noether-type theorems for the nonconservative calculus of variations, asserting that existence of a symmetry implies the existence of a preserved quantity along the solutions of the nonconservative Euler-Lagrange equations (10) – see [3, 4]. Next theorem extends the mentioned Noether-type results: to each symmetry of the optimal control problem (P) there exists a nonconservative conservation law in the sense of Definition 21. Essentially, for $Q = 0$ one gets from Theorem 22 the results obtained in [2, 5, 10, 11]; restricting ourselves to the problems of the calculus of variations, one gets the results found in [3, 4].

Theorem 22 (Nonconservative Noether's Theorem) *If* $(\tau, \xi, \sigma, \alpha)$ *is a symmetry of the optimal control problem* (P)*, and there exists a function* $f = f(t, q, u)$ *such that*

$$\frac{df}{dt} = Q \cdot (\xi - \tau \dot{q}) \,, \tag{18}$$

where Q *denotes the external nonconservative forces acting on the system, then*

$$C(t, q, u, p) = \mathcal{H}(t, q, u, p)\tau - p \cdot \xi + f(t, q, u) - \Lambda(t, q, u, p) \tag{19}$$

is a nonconservative constant of motion.

Remark 23 *Similarly to [2, 5, 10, 11], only the generators* τ *and* ξ*, corresponding to the transformations of the time and state variables, appear in Noether's conservation laws (cf. expression* (19)*).*

Proof. Substituting (cf. Proposition 19)

$$\tau \frac{\partial \mathcal{H}}{\partial t} + \dot{\tau}\mathcal{H} - p \cdot \dot{\xi} = \frac{d}{dt}(\mathcal{H}\tau - p \cdot \xi) - \tau Q \cdot \frac{\partial \mathcal{H}}{\partial p} + \dot{p} \cdot \xi$$

into (14), and using conditions (5), (15), and (18), we obtain successively:

$$0 = \frac{d}{dt}\left(\mathcal{H}\tau - p \cdot \xi - \Lambda\right) - \tau Q \cdot \frac{\partial \mathcal{H}}{\partial p} + \left(\frac{\partial \mathcal{H}}{\partial q} + \dot{p}\right) \cdot \xi \,,$$

$$0 = \frac{d}{dt}(\mathcal{H}\tau - p \cdot \xi - \Lambda) + \xi \cdot (-\dot{p} + Q + \dot{p}) - \tau \dot{q} \cdot Q \,,$$

$$0 = \frac{d}{dt}(\mathcal{H}\tau - p \cdot \xi - \Lambda) + Q \cdot (\xi - \tau \dot{q}) = \frac{d}{dt}(\mathcal{H}\tau - p \cdot \xi + f - \Lambda) \,.$$

■

Remark 24 *Proof of Theorem 22 is very simple. This shows, in the opinion of the authors, that the Hamiltonian formalism provides the natural language to Noether's theory. It is strange, from the mathematical point of view, why the Hamiltonian approach is not used in Physics with respect to nonconservative systems.*

As corollaries, we obtain the previous results known in the literature.

Corollary 25 ***(Optimal Control version of Noether's Theorem – cf. e.g. [2, 10])*** *In the absence of nonconservative forces (i.e. in the conservative case $Q = 0$), if the optimal control problem* (P) *is invariant under the one-parameter transformations* (11)*, then function*

$$C(t, q, u, p) = \mathcal{H}\tau - p \cdot \xi - \Lambda \tag{20}$$

is preserved along any Pontryagin extremal.

Proof. When $Q = 0$, (18) implies that f is a constant, and the conservation law associated with the constant of motion (19) is equivalent to the one associated with (20). ∎

Corollary 26 ***(Nonconservative Noether's theorem of the calculus of variations – cf. [4])*** *For the fundamental problem of the calculus of variations* (2) *the constant of motion* (19) *is equivalent to*

$$C(t, q, \dot{q}) = \tau \left(L - \dot{q} \cdot \frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial \dot{q}} \cdot \xi - f + \Lambda \,, \tag{21}$$

that is, under the invariance hypotheses of Theorem 22, expression (21) *is preserved along all the solutions $q(\cdot)$ of the nonconservative Euler-Lagrange equations* (10)*.*

Proof. For the fundamental problem of the calculus of variations, $\varphi = u$ and $p = \frac{\partial L}{\partial \dot{q}}$, so that the Hamiltonian takes the form $\mathcal{H} = -L + \frac{\partial L}{\partial \dot{q}} \cdot \dot{q}$. Substituting this expression in (19), we obtain the desired conclusion. ∎

Corollary 27 ***(Nonconservative Noether's theorem for problems of the calculus of variations with second-order derivatives – cf. [3])*** *For the higher-order problem of the calculus of variations* (3)*, the constant of motion* (19) *is equivalent to*

$$L\tau + \left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\right) \cdot (\xi_0 - \dot{q}\tau) + \frac{\partial L}{\partial \ddot{q}} \cdot (\xi_1 - \ddot{q}\tau) - f + \Lambda \,. \tag{22}$$

Proof. For the problem of the calculus of variations with second-order derivatives, one has $\mathcal{H}\left(t, q^0, q^1, u, p^0, p^1\right) = -L(t, q^0, q^1, u) + p^0 q^1 + p^1 u$,

$q^0(t) = q(t)$, $q^1(t) = \dot{q}(t)$, $u(t) = \ddot{q}(t)$. Using these equalities, it follows from the Pontryagin Maximum Principle that

$$\begin{gathered}\frac{\partial \mathcal{H}}{\partial u} = 0 \Leftrightarrow p^1 = \frac{\partial L}{\partial \ddot{q}}\,,\\ \dot{p}^0 = -\frac{\partial \mathcal{H}}{\partial q^0} = \frac{\partial L}{\partial q}\,,\\ \dot{p}^1 = -\frac{\partial \mathcal{H}}{\partial q^1} \Leftrightarrow p^0 = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\,.\end{gathered}$$

In this case, the constant of motion (19) takes the form

$$C = \mathcal{H}\tau - p^0 \cdot \xi_0 - p^1 \cdot \xi_1 + f - \Lambda\,,$$

and substituting $\mathcal{H}$, p^0 and p^1 by its expressions, the intended result is obtained. ■

4 Examples

We now illustrate the application of our result to some concrete problems of nonconservative classical mechanics. In all the examples, one can use the computational tools [6] to determine the symmetries.

Example 28 *(Forced Oscillations – cf. [8, pp. 114–115]). Let us consider the problem of vertical oscillations of a body of mass m, connected to a spring of ignorable mass with constant of elasticity k, under the action of a sinusoidal nonconservative force $Q(t) = Fe^{i\omega t}$, where F and ω are positive constants. In this situation, the associated variational problem is given by (cf. [8, pp. 114–115])*

$$\begin{gathered}I[q(\cdot), u(\cdot)] = \frac{1}{2}\int_0^T (mu^2 - kq^2)dt \longrightarrow \min\,,\\ \dot{q}(t) = u(t)\,.\end{gathered}$$

The problem exhibits the following exact symmetries: $(\tau, \xi, \sigma, \alpha) = (c, 0, 0, 0)$, where c is an arbitrary constant. From Theorem 22 we conclude that

$$-\frac{1}{2}\left(mu(t)^2 - kq(t)^2\right) + p(t)u(t) - \int \dot{q}(t)Fe^{i\omega t}dt = const\,, \quad t \in [0, T]\,, \tag{23}$$

is a nonconservative conservation law. Corollary 26 permits to write the conclusion (23) *in the notation of the calculus of variations:*

$$C(t, q, \dot{q}) = \frac{1}{2}(m\dot{q}^2 + kq^2) - \int F\dot{q}e^{i\omega t}dt$$

is a nonconservative constant of motion (nonconservative extremals are defined by $u(t) = \dot{q}(t)$*,* $\dot{p}(t) = F\mathrm{e}^{iwt} - kq(t)$*,* $p(t) = mu(t)$*). In this simple example, such fact is easily verified by direct application of Definition 21:* $\frac{dC}{dt} = 0 \Leftrightarrow \dot{q}\left(m\ddot{q} + kq - Fe^{i\omega t}\right) = 0$*, which is a truism along the nonconservative Euler-Lagrange extremals – the solutions of the equation* (10) $m\ddot{q} + kq - Fe^{i\omega t} = 0$.

Example 29 ([4]) *We apply our results to the example studied in [4]: a dynamical system with Lagrangian* $L(t, q, \dot{q}) = \dot{q}^2/2$*, subjected to the nonconservative force* $Q = \dot{q}^2$*. The associated variational problem is given by*

$$I[q(\cdot), u(\cdot)] = \frac{1}{2}\int_0^T u^2 dt \longrightarrow \min ,$$
$$\dot{q}(t) = u ,$$

and from Theorem 11 we get the exact symmetries

$$(\tau, \xi, \sigma, \alpha) = (2c_1 t + c_2, c_1 q + c_3, -c_1 u, -c_1 p) ,$$

where c_i*,* $i = 1, 2, 3$*, are constants. From Theorem 22, we obtain the nonconservative conservation law*

$$\begin{aligned} &(c_1 q(t) + c_3)\, p(t) + \left(\frac{1}{2}u(t)^2 - p(t)u(t)\right)(2c_1 t + c_2) \\ &\quad + \int \left((2c_1 t + c_2)\dot{q}(t) - c_1 q(t) - c_3\right) u(t)^2 dt = const . \end{aligned} \tag{24}$$

The equation of motion is defined by $u(t) = \dot{q}(t)$*,* $\dot{p}(t) = u(t)^2$*,* $p(t) = u(t)$ *(which is equivalent to the nonconservative Euler-Lagrange equation* (10)*:* $\ddot{q}(t) = \dot{q}^2$*), and we obtain the nonconservative conservation law* (24) *in terms of the calculus of variations* (21)*:*

$$\begin{aligned} &c_1\left[q(t)\dot{q}(t) - t\dot{q}(t)^2\right] - \frac{1}{2}c_2\dot{q}(t)^2 + c_3\dot{q}(t) \\ &\quad + \int \left[2c_1 t\dot{q}(t) - c_1 q(t) + c_2\dot{q}(t) - c_3\right]\dot{q}(t)^2 dt = const . \end{aligned}$$

Similarly to Example 28, also here it is possible to verify the validity of (24) *by definition: the nonconservative extremals* $(q(\cdot), u(\cdot), p(\cdot))$ *are given by* $q(t) = k_1 - \ln(t - k_2)$, $u(t) = \frac{1}{k_2 - t}$, $p(t) = \frac{1}{k_2 - t}$, *where* k_1 *and* k_2 *are constants to be determined from the boundary conditions, and substituting the extremals in* (24)*, we obtain the tautology* $0 = const.$

To finish the illustration of our methods, we consider a generalized mechanical system with one degree of freedom, whose Lagrangian and nonconservative force depends on higher-order derivatives.

Example 30 ([3]) *The following problem is borrowed from [3, §4]:*

$$\begin{gathered} L = \frac{1}{2}\left(\ddot{q}^2 + a\dot{q}^2 + bq^2\right) , \\ Q = \mu\dot{q} + \left(\frac{\mu}{a}\right)^2 \ddot{q} - 2\left(\frac{\mu}{a}\right) \dddot{q} , \end{gathered}$$

where a*,* b*, and* μ *are constants. Replacing expression of* L *and* Q *into* (22)*, we conclude that Noether's nonconservative constants of motion have the form*

$$\begin{multline} C(t, q, \dot{q}, \ddot{q}, \dddot{q}) = \frac{1}{2}\tau\left(\ddot{q}^2 + a\dot{q}^2 + bq^2\right) + (a\dot{q} - \dddot{q})(\xi_0 - \dot{q}\tau) \\ + \ddot{q}(\xi_1 - \ddot{q}\tau) - f + \Lambda , \quad (25) \end{multline}$$

where $f = \displaystyle\int (\xi - \dot{q}\tau)\left(\mu\dot{q} + \frac{\mu^2}{a^2}\ddot{q} - \frac{2\mu}{a}\dddot{q}\right) dt$*. From the necessary and sufficient condition of invariance* (14)*, one obtains that the exact symmetries* $(\Lambda = 0)$ *for the problem are given by* $(\tau, \xi_1, \xi_2, \sigma, \alpha_1, \alpha_2) = (c, 0, 0, 0, 0, 0)$*, where* c *is an arbitrary constant, and from* (25) *we conclude that*

$$\frac{1}{2}\left(\ddot{q}^2 + a\dot{q}^2 + bq^2\right) - (a\dot{q} - \dddot{q})\dot{q} - \ddot{q}^2 + \int \dot{q}\left(\mu\dot{q} + \frac{\mu^2}{a^2}\ddot{q} - \frac{2\mu}{a}\dddot{q}\right) dt$$

is a conserved quantity for the nonconservative system. This conclusion is nontrivial, and difficult to verify directly from the definition of nonconservative constant of motion (Definition 21).

The examples show how the previous nonconservative results found in the literature are easily covered by Theorem 22. Theorem 22 is, however, more general, since it covers an arbitrary complex dynamical control system of the form $\dot{q}(t) = \varphi(t, q(t), u(t))$. Moreover, Theorem 22 introduces a new Hamiltonian perspective to nonconservative Noether's theory.

Acknowledgments

GF acknowledges the financial support of IPAD (Portuguese Institute for Development); DT the support from the Control Theory Group (cotg) of the Centre for Research in Optimization and Control of Aveiro University.

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